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In view of the materially increased costs of all factors that enter into war-time publishing enterprises, and in order that NATIONAL MATHEMATICS MAGAZINE may be freed from embarrassing handicaps; in order that the Administration of the MAGAZINE shall be in a position to publish as rapidly as possible the gratifying number of papers which the fine activity of our Committee Chairmen has committed to our files; finally, in order that we shall be able once more to publish an amount of standard material that shall be not less than 56 pages in each issue of the MAGAZINE we are here announcing again new subscription rates—these having already been announced in the April issue.

I. For all subscription periods beginning with the October, 1944, or a later, issue of NATIONAL MATHEMATICS MAGAZINE the subscription price to all subscribers shall be \$3.00, if the remittance for the subscription is received at the Baton Rouge office AFTER October 1, 1944.

II. If the remittance to cover a subscription period beginning with the October, 1944, or a later, issue of the MAGAZINE is received at the Baton Rouge office BEFORE October 1, 1944, the amount remitted may be based on the present, or old, subscription rate of \$2.00 per year.

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S. T. SANDERS.

The Geometry of the Circular Horn Triangle

By EDWARD KASNER and AIDA KALISH
Columbia University

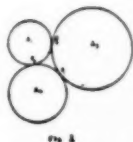
The fundamental ideas for this paper arose during Professor Kasner's lectures in a kindergarten where the youngsters, one day, covered the floor with a thousand tangent, non-overlapping pennies. From this general notion of the packing of coins, the study of the "gaps" or circular horn triangles formed by packing circles in the plane began.*

It is interesting to note that circular horn triangles (Fig. 1) arise in countless ways in our everyday lives—in setting dishes on the table; in the cross sections of pipe lines and cables; and in the human anatomy. In the extension of this idea to three dimensions and the close packing of spheres, it is found that the "gaps" or porosity plays a very important rôle in the mining of oil and in other geological problems.

We shall discuss here only the circular triangle with horn angles. Where we write "circular triangle" we shall mean "circular horn triangle". By definition, a circular horn triangle (Fig. 1) is the triangle formed by three mutually and externally tangent circles. The circular triangle formed by the circles A_1, A_2, A_3 (often referred to as the "given circles") in Figure 2 is the triangle whose sides are the arcs $\widehat{P_1P_2}, \widehat{P_2P_3}, \widehat{P_3P_1}$ and whose vertices are the points of contact P_1, P_2, P_3 of the given circles. As the angles of the circular triangle are horn angles, they are equal to 0° .

Some of the properties of circular triangles that are stated here are metric and some are inversive in character, particularly Theorem 11 which is one of the main theorems of the paper. Also, a few of the theorems are the direct analogues of theorems on rectilinear triangles. For the sake of brevity, most of the proofs have been omitted. Several of the proofs, however, are based on the well-known properties of rectilinear triangles, and, in some cases, the methods of inversion were used.

*Kasner, Comenetz, and Wilkes, *Scripta Mathematica*, March, 1943. Kasner and Supnick, *Proceedings of the National Academy of Sciences*, December, 1943. These deal with the configurations we call the "circlex" and the "hypercirclex" and the proof that the limit of the covering is unity.



Theorem 1. The common internal tangents of the circular triangle are concurrent at the radical center, P , of the given circles (Fig. 3).

Definition 1. The perpendicular bisector of a side of a circular triangle is the line drawn perpendicular to the tangent at the mid-point of that side. The perpendicular bisectors of the circular triangle are the perpendicular bisectors of the rectilinear triangle $P_1P_2P_3$; they are also the angle bisectors of the triangle of centers of the given circles.

Theorem 2. The perpendicular bisectors of the sides of a circular triangle are concurrent at P (the radical center); P is the center of the circle circumscribed about the circular triangle. (Fig. 4).

Theorem 3. The circumcircle of the circular triangle is the inscribed circle of the triangle of centers of the given circles; it is orthogonal to the three given circles and is the only such circle. (Fig. 5).

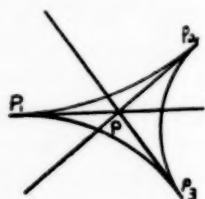


FIG. 3

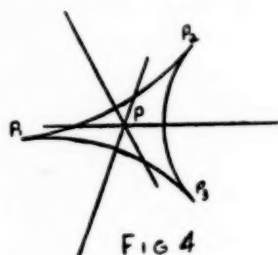


FIG. 4

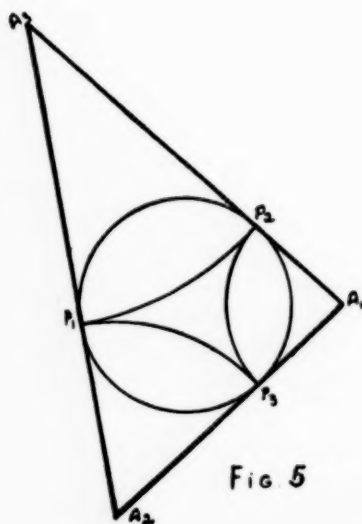


FIG. 5

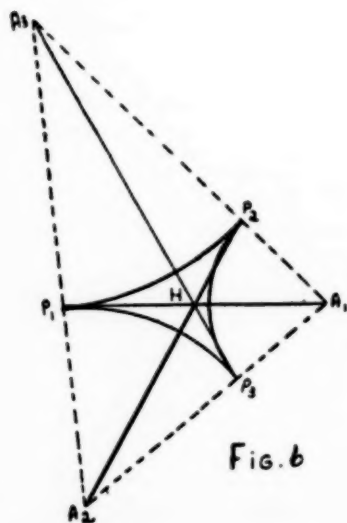


FIG. 6

Definition 2. An altitude of a circular triangle is the line drawn from one of its vertices orthogonal to the given circle opposite that vertex. An altitude passes through the center of the given circle opposite the vertex from which it is drawn.

Theorem 4. The altitudes of a circular triangle, the lines A_iP_i , are concurrent at a point H called the orthocenter. (Fig. 6).

Theorem 5. Any three non-collinear points which form an acute triangle, are the vertices of a uniquely determined circular triangle. Also, any three non-collinear points are the centers of three given circles which uniquely determine a circular triangle.

Theorem 6. The radius, τ , of the circle inscribed in the circular triangle, where a_i are the radii of the given circles, is given by the Steinerian formula

$$\tau = \frac{a_1 a_2 a_3}{a_1 a_2 + a_2 a_3 + a_1 a_3 + 2\sqrt{a_1 a_2 a_3 (a_1 + a_2 + a_3)}} .^*$$

Corollary: The curvature δ of the inscribed circle of the circular triangle, where δ_i are the curvatures of the given circles is

$$\delta = \sum \delta_i + 2\sqrt{\sum \delta_i \delta_j}$$

Theorem 7. The radius, R , of the circle circumscribed about the three given circles, whose radii are a_i , is

$$R = \frac{a_1 a_2 a_3}{a_1 a_2 + a_2 a_3 + a_1 a_3 - 2\sqrt{a_1 a_2 a_3 (a_1 + a_2 + a_3)}} .$$

Corollary 1. $\frac{1}{\tau} + \frac{1}{R} = 2 \left(\frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} \right)$

Corollary 2. If $\frac{1}{R} = \Gamma$, then $\delta + \Gamma = 2 \sum \delta_i$.

Theorem 8. The radius, R' , of the circumcircle of the circular triangle, where a_i are the radii of the given circles, is

$$R' = \sqrt{\frac{a_1 a_2 a_3}{a_1 + a_2 + a_3}} .$$

*Cf. *Mathematical Monthly*, edited by Runckle, Vol. 2, 1860, p. 121. *Nouvelles Annales de Mathématiques*, Vol. 19, 1860, p. 445.—Editor's note.

Theorem 9. The inscribed and the circumscribed circles of the circular triangle intersect at a constant imaginary angle θ such that $\cos \theta = 2$, where

$$\theta = i \operatorname{Log}(2 \pm \sqrt{3}).$$

Proof. Invert the three given circles A_1, A_2, A_3 and the circular triangle $P_1P_2P_3$ with respect to a circle having for center a vertex of the latter triangle, say P_1 .

The circles A_2, A_3 invert into two parallel lines tangent to the inverse A'_1 of the circle A_1 at the points P'_2, P'_3 ; the line $P'_2P'_3$ is the inverse of the circumcircle of the circular triangle $P_1P_2P_3$, while the inscribed circle, S , of that triangle inverts into a circle S' tangent to

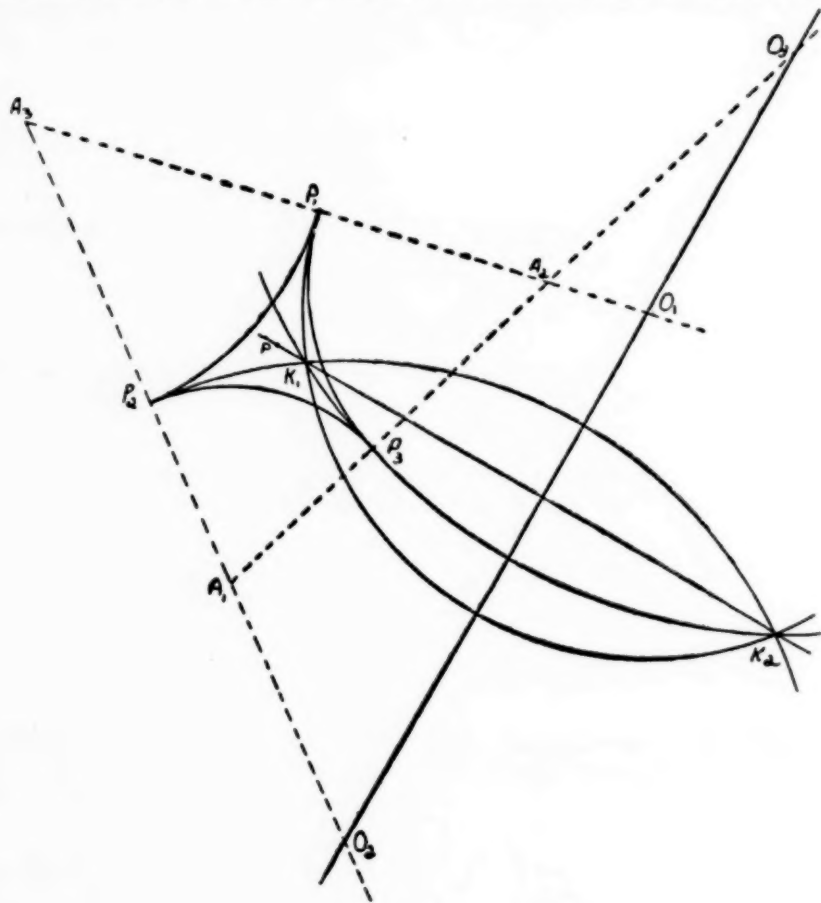


FIG. 7

the two parallel lines considered above and to the circle A'_1 , hence S' is equal to A'_1 .

If the radius of S' is equal to 1, the distances of the center of S' from the line $P'_2 P'_3$ is equal to 2, hence $\cos \theta = 2$, and from the relation $e^{-i\theta} = \cos \theta - i \sin \theta$ we obtain the stated result.

Theorem 10. A necessary and sufficient condition that three circles O_1, O_2, O_3 with radii $\overline{O_i P_i}$, passing through the vertices of the circular triangle and having their centers on the respective lines of centers of the given circles, be coaxial, is that their centers, O_1, O_2, O_3 be collinear. (Fig. 7).

Corollary 1. Circles O_1, O_2, O_3 form a pencil passing through points K_1 and K_2 . Their common radical axis, $\overline{K_1 K_2}$, passes through the point P . When the line $\overline{O_1 O_2 O_3}$ intersects the extended lines of centers of the given circles beyond A_1, A_2, A_3 , the circles O_1, O_2, O_3 are concurrent at point K_1 within the circular triangle. (Fig. 7).

Corollary 2. Circles O_1, O_2, O_3 are concurrent if and only if the product of the cross-ratios $(A_2 A_3 P_1 O_1)$, $(A_3 A_1 P_2 O_2)$, $(A_1 A_2 P_3 O_3)$ is equal to -1 .

Definition 3. The angle bisector* of any horn angle is a circle

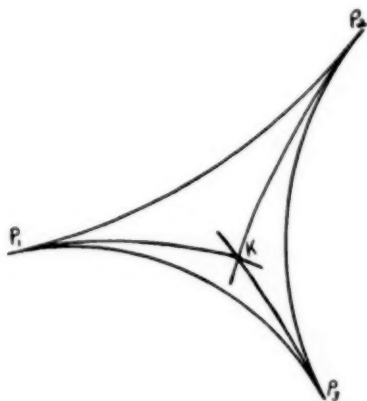


Fig. 8

*The general definition of the concept of bisecting any horn angle (formed by analytic arcs) was given in Kasner, *Proc. Cambridge Inter. Congress 1912*. In the present case the sides are circles and the bisector is a circle. In the general case, use is made of Schwarzian reflexion or conformal symmetry. In the circular case, the appropriate symmetry reduces to inversion. The bisecting circle of the two given circles is the unique circle which, if used as base, would render the given circles mutually inverse.

(passing through the given vertex in the given direction) whose curvature is equal to one-half the algebraic sum of the curvatures of the two given circles forming that angle. It is described so that it is internally tangent to that side of the angle whose curvature is greatest.

Theorem 11. The angle bisectors, B_1, B_2, B_3 , of the circular triangle are concurrent at a point K , within the triangle, which we call the Inversive Center. The angle bisectors intersect each other at K at angles of 120° . They are also orthogonal to the given circles opposite the bisected angles. (Fig. 8).

Many of the theorems of the present paper can be extended to three or more dimensions. In particular the Steinerian formula has been extended by Kasner to space of n dimensions and will be published elsewhere.

EDITORIAL NOTE. Horn-Geometry applied to the circular triangle brings a new approach and a fresh emphasis to a figure which, in itself, is by no means new. It is therefore to be expected that most of the elements which horn-geometry associates with the circular triangle play a rôle in the older geometry as well.

Thus, the orthocenter H (Th. 4) is the Gergonne point of the triangle $A_1A_2A_3$ as well as the Lemoine point of the rectilinear triangle $P_1P_2P_3$.

The bisectors B_1, B_2, B_3 (Def. 3) are the external circles of anti-similitude of the circles A_1, A_2, A_3 taken in pairs; they are also the Apollonian circles of the rectilinear triangle P_1, P_2, P_3 and therefore intersect each other at 120° angles; the inversive center (Th. 11) is one of the isodynamic points of the rectilinear triangle $P_1P_2P_3$, etc.—N. A. C.

Humanism and History of Mathematics

Edited by

G. WALDO DUNNINGTON and A. W. RICHESON

Mathematics in Scotland Before the Eighteenth Century*

By E. R. SLEIGHT

Albion College

A study of the development of mathematics in Scotland centers around the development of the Scottish Universities. However, outside of these great institutions we find many individuals who interested themselves, and the story of early arithmetic is told in the language of "the fastidious and pedantic accountant, the obscure but gifted country dominie, and the village minister with mathematics for a hobby."† These men were not gifted mathematicians, but rather they wrote for the common people.

Any sustained interest in mathematics previous to the founding of St. Andrews University in 1411 was directly under the control of the church, as is shown by the records of the various institutions. The earliest mathematics book mentioned in these records is found in the Glasgow Cathedral list. This book was one of a set on the *Origins of Etymologies*, written by Isidore, Bishop of Seville, (A. D. 570-636). The set of books is an encyclopedia, covering the entire field of learning. Book III contains the subject of arithmetic, geometry, music and astronomy, commonly referred to as the Quadrivium.

Only two pages of this book were devoted to arithmetic. The material was concerned with "fascinating but rather useless excrescences involved in the study of numbers."‡ The type of question would be very simple these days even for pupils in the grades. But it must be remembered that the abacus and the fingers were the only means of reckoning in Isidore's time. For this reason they furnished a certain amount of intellectual training. To a great extent the prob-

*Read at meeting of Michigan Section of Mathematical Association of America, March 18, 1944.

†Wilson: *The History of Mathematical Teaching in Scotland*, p. 1.

‡*Ibid.*

lems involved classification of numbers according to some common properties such as odd or even, powers of two, divisible by two and not by four. The search for perfect numbers seemed to have interested the developing mathematician of that period. In addition to classifying numbers, there is to be found a brief discussion of ratio and the calculation of arithmetic, geometric and harmonic means of two given numbers.

In the geometry covering one-half of a page an attempt was made to define some of the elements of plane and solid figures. Many of these were inaccurate.

Finger reckoning reached its height during the 7th century A. D.* According to this system numbers were distinguished as:

1. Digital, counted on 10 fingers.
2. Articulate, counted on joints of fingers, and forming multiples of ten.
3. Compound, those obtained by combining 1 and 2.

A famous scholar monk, Bede the Venerable, (A. D. c 672) greatly extended the system. "The greatest name in the literature of Saxon England" was applied to him.† He lived in Northern England near the Scottish border. His influence crossed into Scotland and greatly influenced the teaching in its monasteries. His text on finger reckoning is the main source of information concerning this method of computation during mediæval times. Numbers up to 10,000 were represented by showing units on the left hand and tens on the right, together with certain inflections of the fingers. He extended the system to include numbers up to 1,000,000. This he did by placing the hand on various parts of the body, the fingers sometimes straight and sometimes bent.

Bede the Venerable is called the Father of English history, in addition to being a great scientist. His *De Temporium* is particularly important. It contains the theory of tides based upon personal observations. He is given credit for having established the mean interval between the moon's meridian passage and the high water following. His scientific knowledge was superior to that of Isidore. Under Bede a relatively high culture was attained in the various monasteries.

The application of mathematics in those early days found expression in the life of the clergy, either for the purpose of computing the date of Easter, or for efficiency in writing and keeping accounts. It is highly probable that the mathematics involved was much too

*System in vogue in Scotland as late as the 17th century.

†*Encyclopaedia Britannica*, 14th edition, Vol. 3, p. 296.

difficult for the period and that Bede and others of his time made use of an almanac showing the date of Easter for a period of years.

During the 14th century the abacus came into use in Scotland, particularly in the keeping of accounts. Merchants found this method of computation very useful.

It then appears that previous to the founding of the first Scottish University, 1411, the knowledge of mathematics was very limited. For years even after this date the monastery schools were the centers of mathematical training. An Italian scholar, Ferrerius, furnished instruction from 1540 to 1543 at the *Monastery of Kinloss*, located near the present city of Inverness, and commanding a view of Ben Wyvis across the Moray Firth. Little survives beyond two fine round headed archways and a few vaults. This famous scholar "resolved before returning to Italy, to preserve to posterity an account of his lectures and the authors on whose works he had predicated."* Two text books received special mention by Ferrerius: *The Arithmetica of Bæthius* and *Praxis Numerandi*, a version of Sacrobasco's standard text, *De Arte Numerandi*.† From these two texts we may well draw the conclusion that the Abbey of Kinloss was far in advance of its age in the quantity and quality of mathematics taught during these early days. The course of instruction included "the elaborate theory of numbers and ratio transmitted from the Greeks through Bæthius."‡ Since arabic numerals were used in the editions of Bæthius, a knowledge of Algorism was necessary, otherwise the student could not read *De Arte Numerandi*.

Under the teaching of Ferrerius, geometry also made some definite progress. It consisted of definitions and theorems (without proof) found in the first four books of Euclid, together with mensuration involving lengths, areas, and volumes; and areas of triangles, quadrangles, and regular polygons, using the theorem of Pythagoras.

Familiarity with *Spera Mundi*, Sacrobasco's text book in astronomy, was quite necessary for a well educated man during the 16th century; a part of this text was quite mathematical and required geometry as a background. Added to this it was necessary that a churchman should be able to determine the date of Easter. With all of this in mind it is quite evident that the Cathedral schools of Scotland furnished a satisfactory training in mathematics previous to the Reformation (1560).

*Wilson: *Op. cit.*, p. 7, copied from John Stuart, *Records of the Monastery of Kinloss*, pre-face, p. xiii.

†See *Mathematics Teacher*, Vol. XXXV, March, 1942, p. 112.

‡Wilson: *Op. cit.*, p. 7.

Although the University of St. Andrews was established in 1411, and the University of Glasgow forty years later, very little attention was paid to mathematics in the early history of these institutions. The text books used were identical with those used in Kinloss Abbey, and hence the training was essentially the same. The only subjects mentioned not offered in the cathedral schools were perspective and geography. Mathematics was optional in the universities, and there is no evidence that there was any great amount of interest in the subject.

From all existing evidence it appears that mathematics was very little used in Scotland until the beginning of the Reformation, about the middle of the 16th century. According to W. Boyd "The Scottish reformers made ample provisions for educating the people in practical Arithmetic."* Parochial schools were founded in which reading, writing, and arithmetic were taught.

New demands for mathematics were created. The development of commercial arithmetic as a result of commerce, and the necessity for keeping accounts, was very marked during this period. Also added interest in teaching resulted from the fact that textbooks were beginning to appear in the English language. Previous to this time all scientific works had appeared in Latin. The first of three books to be written in the English language is credited to Robert Recorde, an Englishman. Four texts, very remarkable for the age, were written by him, and appeared about the time of the Reformation. Some of these were very popular in England, and there is some evidence that they influenced the teaching of Arithmetic in the Scottish schools.

Multiplication and division were facilitated by use of Napier's Rods, or Napier's Bones, as they were called in those days. Mechanical devices for performing the fundamental operations were very popular in England, and these were widely used, especially among the common people. The scientists think of Napier as the inventor of logarithms, but his fellow countrymen of the 17th century found too many difficulties in their use, and to them Napier's fame rested upon his invention of the system of computation known as Napier's Rods.

The development of mathematics in the Universities during this same period was greatly encouraged by Andrew Melville, who was appointed Principal of Glasgow University in 1574. Melville "taught the Elements of Euclid with Arithmetic and Geometry of Ramus,"† a brilliant French Mathematician who lectured at the University of Paris about the middle of the 16th century. He was quite revolutionary in his thinking. At one time he attracted con-

*W. Boyd: *History of Western Education*, p. 209.

†Sir Alexander Grant, *The Story of the University of Edinburgh*, Vol. 1, p. 82.

siderable attention with a declamation which he delivered on the thesis that all Aristotelian philosophy was false. He was an outstanding orator and a debater. His brilliant career ended on St. Bartholomew's Day, August 26, 1572.

Since the algebra and geometry of Ramus were prescribed by the Senate of the University of Glasgow as two of its texts, we may learn something about the mathematical training in that institution during the latter part of the 16th century. The algebra of Ramus was divided into two parts, *Numerato* and *Aequatio*. The former dealt with addition, subtraction, multiplication, division and root extraction. He made use of the positive and negative signs, but he used no letter to designate the variable. He did use letters to indicate powers; l for the first power, q for the second, c for the third, bq for the fourth, and others letters and combination of letters for higher powers. According to his notation $4x^3 - 3x^2 + 2x$ was written $4c - 3q - | - 2l$. Some letters were used with two meanings. This is true of the letter l , which was also used to indicate square root. Thus the $\sqrt{2}$ was written $l2$. It is quite probable that this letter was suggested by the word *latus*, the Latin for side, the $\sqrt{2}$ representing the side of a square with an area of 2 units. This notation presented difficulties when a coefficient was used. For this reason $3\sqrt{3}$ was written $l27$.

Aequatio (Algebra) was used by Ramus mainly for the purpose of solving equations. He was able to solve any problem, provided the equations contained one unknown only, and was linear or quadratic. If the equation was of a higher degree, then his ability extended only to those forms in which only one unknown term appeared.

The sexagesimal system was used by scientists during this period; for this reason Ramus knew nothing about decimal fractions. A properly constructed sexagesimal system requires a multiplication table up to and including 59×59 , all products being expressed in terms of the base 60. His text employed the four fundamental operations, and when we consider the possible errors in operating such a system it is no wonder that Kepler was prompted to make the statement "Napier has conferred a great benefit upon Astronomy," when Napier's system of logarithms first appeared.

The geometry of Ramus included 27 books, and was considered by him a study in "earth measurements." He collected all the known theorems concerning the triangle in one book, those of the circle in another, etc. His aim was to arrange all the known facts of geometry in the most convenient form for the surveyor and gauger.

The geometry of Ramus was in reality, a textbook on practical mathematics. "Though this was one of the books prescribed by the Senate of the University of Glasgow during the reign of James VI, it

is highly doubtful if the standard set by the book was attained by the students. In view of the fact that little or no arithmetic was taught in the schools and that in the universities it formed only a part of one year's course, it is hard to believe that the graduates could have possessed all the *Arithmeticae et Geometriae* of Ramus."* Even though the students did not reach the standards set by Melville, it is very certain this influence did much toward the development of mathematics in Scotland during that early period.

When Melville's years of service at the University of Glasgow ended, a reaction against his program set in, and there was very little interest in mathematics in that institution until nearly a century later. At the beginning of the 17th century Recorde's arithmetic† under the title of *The Ground of Artes* appeared and was very widely used in England. However, as previously stated, its influence was not great across the border, as is shown by the fact that arithmetic was very little known. Even students who were prepared to enter the University could "hardly find the number of the pages, sections, chapters, and other divisions in their book, to find what they should."‡

A study of the parochial and grammar schools, as well as the commercial academies reveals the fact that this indifference to the study of arithmetic was not changed until the latter part of the century. With the exception of the parochial, the curricula of these various types of schools make no mention of its study until 1660. Even here we are forced to believe that its study must have been very meager, in view of the fact that it was taught in the universities in a very elementary way. Later in the century, however, these same elementary schools included the fundamental operations in integers and vulgar fractions, but with the development of foreign trade a greater knowledge of arithmetic was demanded. This, together with the fact that textbooks were beginning to be written in English, greatly increased the interest in the subject.

In 1678, Cocker's arithmetic first appeared.§ Although written by an Englishman, it was used very extensively in the schools of Scotland. Cocker's book passed through many editions, and for better than a century was the standard text in both England and Scotland. He taught the subject by a "Simple Method suited to the meanest capacity."

*Wilson: *Op. cit.*, p. 18. Copied by Wilson from Thomas McCrie, *The Life of Andrew Melville*, p. 239.

†*Early English Arithmetics*, NATIONAL MATHEMATICS MAGAZINE, Vol. XVI, February, 1942, p. 243.

‡Wilson: *Op. cit.*, p. 30. Quoted from John Brinsley, *Ludus Literarius on the Grammar Schoole*, p. 25.

§NATIONAL MATHEMATICS MAGAZINE, Vol. XVII, March, 1943, p. 248.

Commercial academies made their appearance during the last two decades of the 17th century. The first of these was established in Edinburgh in 1680, followed soon after by a second school of the same type in Glasgow. The textbook used was written by James Paterson under the title of *Scots Arithmetician*. The type of arithmetic taught, and the stage of advancement in this subject at that time may be determined by noting its contents. Napier's Rods were still in use as an aid to multiplication. It appears, however, that the pupil was compelled to learn the multiplication table, a procedure which did not last long, as most of the texts of that day, as well as those of a century later contained the multiplication table as an "aid to the memory."

Vulgar fractions were discussed in Paterson's text,* but he used a symbolism quite unlike that of the present day. Three systems are found:

$$\begin{array}{c} N \quad D \\ 3 \quad 4 \end{array} , \frac{D_3}{D_4} ,$$

and $3\frac{3}{4}$ all represented $\frac{3}{4}$. In writing decimals we find the colon, the inverted comma, and the right angle. Thus:

$$:3 = \frac{3}{10} , :002 = \frac{2}{100} , '05 = \frac{5}{100} , 25\angle 4 = 25\frac{4}{10} ,$$

Each new rule was introduced by doggerel verse. Thus for multiplication:

"To learn a-right to Multiplie
The table get in memorie
Then set down multiplicand
And nixt, let multiplier stand.
Multiplicand then Multiplie
First place of Multiplier by
Units set down; tens keep in minde
To add when ye occasion find"—etc.

Paterson's text offered a fairly adequate training in arithmetic, in spite of its peculiarities. The fact that it was written in English by a Scotsman was greatly to its advantage.

So far, we have noted that very little mathematics was taught in the elementary schools until late in the 17th century. This was due in part to the fact that arithmetic and elementary mathematics were not studied until a student entered the university. In 1648 a commission was appointed to examine the course of study in the various

*Wilson: *Op. cit.*, p. 27.

universities. In St. Andrews the four year course was divided as follows:*

First Year: October until November, Latin; November to June, Greek; the remanent time of that year after the month of June to be spent in learning the elements of the Hebrew tongue that at last they may be able to read the elements of Arithmetic, the four species at least.

Second Year: Logic and Rhetoric.

Third Year: The elements of Geometry and the first and second books of the Arithmetic.

Fourth Year: The other four books of the Arithmetic, the De Cælo, elements of Astronomy and geography, De Ortu et Interitu, the Meteors and De Anima.

The amount of mathematics taught in Glasgow University during this period is indicated by the text used. It was written in Latin by George Sinclair and published in 1661 under the title of *Tyrocinia Mathematica*. This book contained all the mathematics taught in this institution. Sinclair was a regent of the University but was deposed soon after the completion of his text. He was reinstated, and in 1691 he became the first Professor of Mathematics in Glasgow University. That his lectures differed very little from those of 1661 is shown by the fact that he used an English translation of *Tyrocinia Mathematica*, which was in nearly every respect indentical with the Latin edition.

The text itself was very simple. The arithmetical part involved only numeration and the four fundamental operations applied to integers. Fractions were completely ignored. To some this might seem very strange in view of the fact that Napier's system of logarithms was coming very rapidly into use. But according to Napier, the functions were defined in terms of line lengths, rather than ratios. As a result, the sines and tangents were large integers, thus making decimals unnecessary. Napier's Rods were used for multiplication, and the results verified by *casting out the nines*. Astronomy followed the few brief pages on arithmetic. Here again very little knowledge of mathematics was necessary, as astronomy had been developed on practical rather than scientific lines. An instrument called the *Sector* was used for solving problems in spherical trigonometry. Measurements of such astronomical data as declination, amplitude, and azimuth also were determined by use of mechanical devices.

Geography and navigation were included in the university course. All practical problems were eliminated, with the exception of the

*Wilson: *Op. cit.*, p. 23.

measurement of distances on the earth. Some spherical trigonometry was required.

In the University of Edinburgh, the youngest of the four such institutions in Scotland, there seems to have been more advancement than in any of the others. This is particularly true in the field of mechanics. This awakened interest in mathematics at the close of the 17th century can be directly attributed to the influence of three members of the Gregory family. The first of these, James Gregory,* received his appointment as Professor of Mathematics in 1674, but died one year later. Like most of the leading mathematicians of that period he was interested in both physics and mathematics. In 1661 he invented the reflecting telescope which bears his name. In the field of pure mathematics he was interested in expansion into a series of such functions as $\tan^{-1}\theta$, $\tan \theta$, and $\sec^{-1}\theta$. Also he distinguished between convergent and divergent series, and proved that π is incommensurable. This latter, though ingenious, was not very rigorous. The series $\theta = \tan \theta - \frac{1}{3} \tan^3 \theta + \frac{1}{5} \tan^5 \theta - \dots$ is often called the Gregorian series.

The second member of this illustrious family to become Professor of Mathematics at the University of Edinburgh was David Gregory, a nephew of James.† He was an authority in the field of geometry, as well as a brilliant teacher. He lectured on trigonometry, logarithms, practical geometry, geodesy, optics, dynamics, and mechanics. He was one of the first lecturers on the Newtonian philosophy and introduced the *Principia* to the students of Edinburgh. Gregory's fame in England led to his removal from the University of Edinburgh, where during nine years, he had "brought the mathematical teaching into the vanguard of scientific progress." In 1692 he was made *Fellow of the Royal Society*, and shortly after was appointed Professor of Astronomy at Oxford.‡

The last of this illustrious family to be elected to the chair of mathematics at this same institution was a second James Gregory, who held this position for 33 years. "He seems to have been an able teacher, but did not otherwise add to the reputation of the Gregory family."¹

As a result of the influence of these three men, interest in mathematics at the University of Edinburgh was greatly increased, and all types of institutions in Scotland felt this influence.

*Sir Alexander Grant, *The Story of the University of Edinburgh*, Vol. II, p. 295.

†Sir Alexander Grant, *Story of the University of Edinburgh*, Vol. II, p. 297.

‡*Ibid.*

¹*Ibid.*, p. 298.

In this article we have endeavored to examine the status of mathematics in Scotland from the earliest known records to the close of the 17th century. In spite of religious wars, coupled with the fact that as a nation there was very little interest in scientific fields, we find that this period closed with an awakened interest, both in the demands for and interest in mathematics. Practical arithmetic was taught in the Parochial Schools as well as in many of the Grammar Schools. Commercial Academies were established in which the study of arithmetic, navigation, and trigonometry was given great attention. Finally in the universities, mathematics was compulsory, and included Euclid, surveying, plane and spherical trigonometry, astronomy, navigation, and algebra up to the solution of the quadratic equation.

At the opening of the 18th century the needs of the nation required mathematics, and there was no longer any doubt as to its importance in the curricula of the various types of schools in Scotland.

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CORRIGENDA

Page 286, last line: in the denominator of the third term within brackets, delete p_1 .

Page 287, first line: in the last term within brackets, for p_n read p_m (two places.)

Page 287, nineteenth line: in the last term within brackets, for the exponent m read n .

The Teachers' Department

Edited by

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Trends in the Teaching of Secondary School Mathematics

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Some twenty years ago there was a notable shortage of teachers of high school mathematics. The recently elevated colleges of education were being taxed to capacity to fill the new openings and to provide for the normal turn-over in the teaching personnel. Unlimited seemed the opportunities for college graduates who had majored in mathematics to find useful and profitable service in the field of secondary school teaching. Retired clergymen, superannuated football coaches, jobless engineers, displaced Latin teachers all flocked in to take the more lucrative positions and still there was room for more, ever more room, in the ranks of teachers of algebra.

The urgent call for teachers of mathematics was but an incident in the extraordinary expansion of the high school population. It was not due to any fresh viewpoints on the philosophy of mathematical education. With the rise of the dogma of American imperialism came a fresh popular outlook manifesting itself in such scattered phenomena as the American summer tourist migrations to Europe, the Chautauqua circle, and a lengthening average period for schooling. The rapid spread of the acceptance of an extended school career for the adolescent introduced many problems not clearly grasped at the start. One problem however was immediate and pressing—to find teachers to handle the classes. And mathematics, which had always been a conspicuous high school subject, shared in this general expansion.

The call for large numbers of new teachers placed emphasis on quantity, and in many school systems quality seemed almost forgotten. But the American parent was demanding that his boy and his girl be given something better than what in former days the little red school house with the single ungraded room had been able to provide. School principals began to feel competitive pride in pointing to a roster of

specialists, and to a proud sprinkling of collegiate degrees among their listed staff. Most universities continued to treat the preparation of teachers as a trivial or at most incidental objective remote from their major aims. Yet many an undergraduate was persuaded to continue to graduate study, purportedly perhaps to prepare for a career of research via a doctoral degree, but actually to secure (and with minimum labor) a master's degree and so to further his chances in the political arena of a municipal school system. As knowledge of European standards became widespread among the outgoing thousands of young teachers versed in the history, theory, and pre-practice of teaching, the public became conscious of the fact that blameless character and effective discipline were compatible with a subject-matter ignorance so profound as not to fool even a dull pupil. In the loud clamor for more teachers, there was to be heard a faint but insistent plea for teachers rather better informed than those needed for "readin' and writin' and 'rithmetic, taught to the tune of a hickory stick." When Regents' examination and College Entrance Board examinations revealed with inescapable harshness the lack of adjustment between a given school program and an external and powerful even if imperfect objective standard, the injured parents would occasionally insist on higher standards of proficiency for teachers. Amid all the hasty adjustments to meet the on-rushing flood of high school enrollments, there was ever the clamor for more and better teachers.

But a profound change in educational outlook took place. The secondary school system became rapidly estranged from century old standards of college preparatory training. The prophets of democracy among educators, urged acceptance of the world challenging doctrine of wholesale secondary school education. No longer do thoughtful Americans accept the idea that a high school diploma is to be the distinguishing badge of the book-worm with high I. Q. Currently it is agreed that the community should afford to practically everyone the economic and physical opportunity to attend school at least through the age of about 18 years; and that correspondingly, it is the duty of the school system to provide training of such a sort that the many hours spent in school may be profitable to all concerned. Not the fear of inescapable failure, nor the drudgery of uselessly repeating a hated and meaningless routine, but the incentive of assured and conscious progress, the exercise of all one's talents, the growth toward self-reliant citizenship, are to be the heritage of youth. In short, the secondary school should provide the natural living-space of the young people of the nation.

The spreading acceptance of an ideal of uninterrupted schooling for the whole adolescent population reacted suddenly and disastrously

upon enrollment in mathematics classes. Dropping from a traditional schedule usually of four years of high school mathematics, there was a rapid downswing until in many secondary schools, the large majority of students avoided mathematics altogether, or at most, took a brief course in commercial arithmetic. Teachers of mathematics were stunned. Many turned to other fields such as pupil-guidance, social studies, or teacher-supervision. Some yielded in disgust or in sorrow; others protested with dire warnings to all who would neglect fundamentals. Many, without attempting to understand the nationwide change of policy, urged the return of disciplinary values. Still others sought in a fumbling, faltering way to present important mathematical principles in so captivating a form as to inspire such effort as is needed to master essential techniques. But, whatever may be said as to the appropriate manner of coping with the changing situation, there could be no question that, by and large, mathematics in the secondary schools was on the defensive and was losing ground. For the energetic student with a demonstrated knack for mathematical reasoning, there might still be in the large schools, special courses in solid geometry and trigonometry. Technical schools even introduced courses in calculus for the eleventh grade. The more ambitious preparatory schools gave better courses than ever in mathematics in the fiercely accelerated competition to enter their graduates in the major colleges. But, whatever its loyal supporters might claim as to the necessity or propriety of mathematics as a basic ingredient in the secondary school curriculum, the fact remained that this subject was slipping rapidly from its long acknowledged preeminence as a major required subject into a minor position as an exclusive elective subject recommended for only those with special aptitude.

Sufficient causes of this decline are fairly obvious. Many thoughtful persons, themselves products of a system of required mathematics, professed to have found little social value in their painful exposure to the subject. Mathematical exercises reveal with brutal clarity the discrepancy between correct and submitted answers. Carelessness in technique, failure to understand, slowness in computational procedure, each individually or in combination, suffice to mark the students' efforts as unacceptable. Repeating the course rarely corrected the true source of the difficulty. If the young person was to progress through school with his own social group, then the unyielding requirements of mathematical proficiency had to be sidestepped by the only means that seemed practicable—that of omitting the subject from the student's list of peremptory assignments. Many prophets of a new day, sensing entrenched interests, went much further and questioned the utility of mathematics even for most of those who liked it. On the

basis of questionnaires, whole sections of arithmetic and algebra were sometimes condemned as anti-social, and worse than futile, save possibly for prospective technicians.

Then unexpectedly, even with crass suddenness, our nation found itself engaged in wholesale preparations for defense. Hardly had these preparations gotten under way before we were drawn into the greatest war of modern times. Our daily petty extravagances faded out under the cloud of rationing. From a decade of business depression and of Federal relief measures, we were thrust into a period of drafted manpower and of limitless opportunities for high wages among the undrafted and untrained. Boys of eighteen, who otherwise might have been wandering the streets trying to find a chance to earn a living, faced death in jungle and desert. While lives are being drafted and earning conscripted, the schools naturally respond by seeking and accepting authoritarian curricula. One does not judge by standards of ultimate civilian utility, nor does one weigh cultural as against economic pressures while a total war challenges our national existence. When those responsible for our national effort approve certain war training courses, the school systems throughout our land proceed to furnish such courses as best they may.

War with its modern emphasis on mobile mechanized equipment, its use of fleets of airplanes, and its employment of innumerable dials and indicators, turned naturally to the engineers and physicists. It became apparent at once that the current civilian supply of technicians was entirely inadequate to our wartime needs. The air forces, the Armored Force, and the Signal Corps were bidding against each other and could not fill their vacancies. Schools and colleges were urged to provide suitable basic courses for handling technical equipment, and with the sponsorship of the U. S. Office of Education, such courses sprang into being all over the country. Along with more specialized technical training, mathematics received a fresh impetus. Algebra, trigonometry, coordinate geometry, vectorial methods, and calculus, became objects of popular demand. So-called "refresher courses" supposedly helped to recall the fundamentals of mathematical computation to those whose scientific training needed development. As teachers of mathematics and physical sciences were drained off to conduct war work, the teaching of these subjects became increasingly important and progressively difficult to administer. From the social and literary studies, teachers were being diverted and reconditioned to teach science and mathematics.

Today we are in the midst of a wartime program. Teachers are scarce, what with the draft, the women's service organizations, the high wages in airplane factories and ship building establishments,

and so forth. Teachers of science and mathematics are particularly in demand. It continues to be the patriotic duty of the schools to provide, at least for the young men nearing their draft age, as broad and efficient technical training as possible. There is little question as to the teachers' immediate duty. But what of tomorrow? Impatience and optimism seize upon attractive bits of news or rumor, and many a seer or dreamer is formulating a prospective post-war policy. With no attempt to predict, one can inquire as to the predictions of others.

Already numerous divergent views are being expressed. In particular, one meets many teachers who feel that despite the temporary inconvenience and confusion of a world struggle, no essential change is to be expected in the teaching of mathematics during the coming post-war era. The old school building will get a cleaning, the rallies for blood banks and for war loan bonds will stop, the victory gardens will relapse into lawns and tennis courts, the old required courses will be reinstated, the old texts used once more. We shall have learned nothing from the war that should alter our view of mathematics. How comfortable it is to be reassured that the next generation will be like the last! Sound educational principles, we are told, are not to be altered by the accident of a passing war. One adopting this complacent view may shut his mind to all educational discussion, and close his eyes to all social phenomena, prejudging such matters to be irrelevant. But may not a generation which has faced death in strange lands, seen empires collapse, and new concepts of world federation emerge, feel free to reinvestigate the principles and practice of education? Can indoctrination and propaganda become household words, and the school curriculum remain untouched?

Remote from these unquestioning conservatives are the self-vindicated prophets of disciplinary virtues who would revive the stern glories of a rigorous mathematics course for all secondary school pupils. School promotion, they say, should be based as of yore upon demonstrated mastery of fundamentals. Better that many should fall by the wayside in a sincere but futile struggle to achieve the art of computation and the logic of demonstrative geometry than that the capable few should waste their time in a course diluted for dullards! If mathematics means sound thinking (and what self-respecting "mathematicians" would deny it?), then surely we must see to it that all our young people have as thorough a course in mathematical drill as the schools can provide. These prophets point with solemn pride to the school discipline of former generations and cite with triumph the demonstrated need for wide-spread rigorous mathematical training, even as so publicly proclaimed by some leaders of our armed forces.

We were wakened from foolish dreams of isolation, from soft courses and interest-centered curricula, these say, by the cruel realities of war. Let us never again forget that eternal vigilance is the price of liberty, and that mental discipline in school is the price of sound thinking among the citizenry. The war, these declare, should have opened our eyes to truths we had begun to forget, and should have stirred us from lazy somnolence. A generation which has forgotten how to chastise its wayward offspring must return to age-old disciplinary values. The Spartan training so essential to military progress must have its normal equivalent in the no less arduous pursuits of self-discipline amidst the enervating influences of peace.

But there will be those who will ask whether the cheerful endurance of the patriot is indeed the normal tribute of peace. Is the regimentation which flings lives into a combat, erases the artist, confiscates private enterprise; is the all-out effort for war to become an end in itself, an objective to be cherished at the sacrifice of the individual? Must this painful interruption to the orderly evolution of sane general education be accepted as an inherent blessing? Are we not to hope once more for a return of freedom, for a restoration of the right of an individual to develop his own powers within the framework of an evolving social structure? To many it would seem that real self-discipline along the lines of one's own powers and personally acknowledged responsibilities will in times of peace act as an incentive, whereas the unreasoning imposition of an arbitrary universal standard of achievement in school, or outside, will often, probably usually, stifle aspiration. One will surely forget most of the routine details which are impressed by drill but never put to practical use. For many persons, ability to handle numerical fractions with speed and certainty has about as little practical or theoretical value as the ability to rattle off a list of deponent verbs in Latin, or the list of states of the Union in alphabetical order each with its area in square miles. With this, let us glance at another opinion.

Our country is firmly resolved to see the present world struggle through to an effective conclusion. But as individuals we can be forgiven for soft nostalgic memories and for fondly dreaming of a new dawn. We are not enamored of fighting on its own account. The more conscious we become of the present need for whole-hearted consecration to the war-effort, the more steadfast grows our determination that values of peace shall not be casually endangered. One may expect a major readjustment when peace is declared. To disband a huge military machine will well entail problems of reemployment, reconditioning, reinvestment, such as the country has not seen for three-quarters of a century at least. In our eagerness then to grapple

with post-war problems, may not we ditch our interim wartime philosophy and reject as outworn some of the pronouncements engendered during the fray? There are a few teachers who expect a war-weary generation to effect an about-face, to hail with feverish eagerness the half-forgotten opportunities for study of art, of literature, of social science, of philosophy. The public, convinced of the need for mathematics in the program of war, as it lays aside interest in military science, may be tempted to shelve a mathematical training beyond arithmetic save for technical students. A bemedaed bombardier's scoffing remark, "Oh, we never had any use for all that high-brow mathematics", is not likely to encourage persistent demand for this subject among those who plan to engage in non-technical work. It may be that any comprehensive program for general mathematical training at the secondary school level will be on the defensive as never before. In any case, let no one blindly assume that the current extensive regimentation in mathematics as in other lines is being accepted as a permanent ideal.

There is a fourth view to which many subscribe. Assuming that the demand for secondary-school mathematics will revert to approximately the previous level, one may still question whether the proverbial type of course and the time-honored method of instruction will continue to satisfy the tax-paying citizenry. The stark reality of the present bitter struggle waged in foreign lands may impress upon the minds of survivors the need for an immediate grasp of essentials, power to think quickly, to improvise, to estimate, to face new situations, and to carry on in the face of obstacles. There may well emerge an impatience with the more remotely implemented topics, an insistence on challenging exercises, a scorn for unrealistic precision. The troops today are being given brief, condensed survey courses, supplemented by simulated combat conditions. Perhaps our school courses in trigonometry have for the majority of students failed to make adequate contact with surveying, astronomy, and navigation. Demonstrative geometry, poorly adapted to a brief exposure, may need motivation in logical directions, as never before. Probably the capsule course, forced on us by the tempo of the war, will be laid aside with relief. Pupils may once more be free to raise questions, to ponder over problems, and to grow gradually with the subject. The vital role of mathematics in the technological phases of our current cultural pattern may need to be emphasized if the post-war teacher is to capture the attention of a war-exhausted people. The national advertisers who are painting such rosy pictures of the wonders of a new day are arousing the imagination of our young people. With a little attentive foresight, teachers can capitalize on this interest, while blind conserva-

tivism can lose many a convert. If mathematics claims attention only as a conventional course organized along traditional lines, the teacher may find few pupils to respond to it. If its practical utility, its beauty, its essential role in interpreting the times becomes clear,—if it can challenge the brightest minds, and yet prove rewarding to the mentally retarded, no one need fear for the mathematical education of the next generation.

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Zero: the Symbol, the Concept, the Number

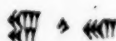
By CARL B. BOYER
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In the history of mathematics considerable confusion exists as to the origin of zero. To the extent that this is the inevitable consequence of lacunæ in the historical data, only the disclosure of further evidence can alter the situation. Nevertheless, there are aspects of the difficulty which can be clarified by the simple expedient of stating the problem unequivocally. There can be no unique answer to the query, "Who first discovered the zero?", for this may refer to any one of several related but distinct historical issues. One of these would seek the earliest use of a *symbol* or mark (cipher) to indicate an empty place in a so-called positional notation for integers and fractions, such as is evident in the number 102.03. One special aspect of this first form of the question would concern specifically the oldest use of zero in a *decimal* positional system; another might refer in particular to the origin of that *type* of symbol which now is employed in all Western civilizations; still another aspect would concern the origin of the *word* zero. A second more general form of the problem would call for the first reference, in connection with the idea of number, to the *concept* of a null class or an absence of magnitude. An answer to this inquiry would depend upon nice distinctions between the philosophical void and the mathematical zero, or between concrete objects and abstract ideas. A third comprehensive form of the query would seek the earliest recognition of zero as itself a *number* subject in general to the ordinary arithmetic operations. To satisfy this question one must know whether an elementary or a more sophisticated concept of number is called for. With the problem stated in these several forms one may well hesitate nonetheless to hazard anything beyond a tentative answer.

The facility with which we now carry out computations is ascribable largely to two principles*—cipherization and local value—both of which were known perhaps 4,000 years ago. In the Egyptian hieratic ciphered numeration, as in all non-positional systems, there was no need for a symbol corresponding to zero. In the positional notation of the Babylonians a symbol for empty places was desirable to avoid

*See my paper, "Fundamental steps in the development of numeration," scheduled to appear shortly in *Isis*.

difficulties. However, in this sexagesimal numeration empty positions occur far less frequently than in the decimal system (not at all for integers less than sixty; and in only 59 cases for integers less than 3600, as compared to 917 in the ten-scale), and contextual ambiguities seldom arose. For this reason the early Babylonians who first developed the positional principle felt no urgent need for a symbol to cover this situation. At some time of about the Persian period, however, a conventional sign, \blacktriangleright , was adopted to mark empty places in their notation.* In accordance with this notation, the number 162,032 would be written as



—that is, as $45(60)^2 + 0(60) + 32$. This represents probably the earliest appearance of a *symbol* for zero, corresponding to our own use of the “goose-egg” in ordinary calculation.

Symbols for zero or for empty places appear also in inscriptions of the ancient Mayan civilization in which a calendrical notation based on local value had been adopted. These lenticular symbols were variously embellished—as



—but the variants were easily recognized as in all cases indicating empty positions. In the Mayan bar-and-dot scheme the symbolism represented $2 \cdot 20^4 \cdot 18 + 0 \cdot 20^3 \cdot 18 + 10 \cdot 20^2 \cdot 18 + 0 \cdot 20 \cdot 18 + 3 \cdot 20 + 0$ or 5,832,060 days.† However, there is no indication that this system, introduced by the Mayas as early as the beginning of the Christian era, was used in general computation.

Babylonian influence in Hellenistic astronomy was so strong that although ordinary Greek numeration for integers was decimal, a sexagesimal scheme of subdivision into fractional parts was adopted in astronomy. In terms of the Ionian alphabetic notation, such a number as 321.3892 would take the form $\tau \kappa \alpha \kappa \delta' \kappa \alpha''$ —that is,

$$321 + \frac{23}{60} + \frac{21}{60^2}$$

*For Babylonian notation see Otto Neugebauer, *Vorlesungen über Geschichte der antiken mathematischen Wissenschaften*, Vol. I, *Vorgriechische Mathematik* (Berlin, 1934); also F. Thureau-Dangin, “Sketch of a history of the sexagesimal system,” *Osiris*, VII, 85-141.

†See S. G. Morley, *An introduction to the study of the Maya hieroglyphs* (Washington, 1915), pp. 129-133; cf. p. 8ff. See also D. E. Smith, *History of mathematics* (2 Vols. Boston and New York, 1923-1925), II, 43-45; Karl Menninger, *Zahlwort und Ziffer* (Breslau, 1934), pp. 39f, 314f; Florian Cajori, “The zero and principle of local value used by the Maya of Central America,” *Science* (N. S.), XLIV, (1916) 714-717.

When in such a sexagesimal fractional form an empty position occurred, Greek astronomers indicated this fact by using the letter omikron (not to be confused with the use of this letter for 70 in the decimal alphabetic notation for integers). This symbol may have been suggested by the first letter of the word *οὐδέν* (empty or void). The number

$$321 + \frac{21}{60^2}$$

would thus be written as $\tau \kappa \alpha \text{ } \circ' \text{ } \kappa \alpha''$. Here, just as in Babylonian and Mayan numeration, one sees clearly the use of zero which corresponds in all respects to our own except that the scale of notation was not decimal.

From the Babylonian, Mayan, and Greek representations the answer to the first general query above appears reasonably clear: a symbol for zero appears to have been used first by the Babylonians, somewhat later by the Greeks and the Mayas. It is to be noted, however, that in these systems the symbol \blacklozenge or \circ or \bullet represented an empty position and nothing more. There is at the present time no evidence that any one of these forms appeared alone, independent of other digits in positional notation. Hence, one may not, without further assurance, interpret these symbols as representing the *concept* or the *number* zero.

Early philosophical references to the "void" may be regarded as implying a similar mathematical conception, and so are related in a broad sense to the notion of zero; but specific statements in this respect were lacking. From the infinitesimal concepts of the early Greek mathematicians one can similarly infer the tacit recognition of an idea corresponding to zero; but here too there was no explicit formulation. To Plato has been ascribed the conception of zero (as well as of negative quantity),* but this attribution is based on somewhat devious arguments derived from the philosopher's recondite terminology. Possibly the earliest clear and explicit reference to the mathematical *concept* of zero is found in the *Physics* of Aristotle. Here the Stagirite had propounded the doctrine that the speed of a body is inversely proportional to the resistance of the medium in which it moves. In considering motion in a vacuum he concluded that a thing would move through the void with a speed beyond any ratio (and hence that the existence of a void is impossible), for "there is no ratio in which the void is exceeded by body, as there is no ratio of zero to a number . . . For this reason, too, a line does not exceed a point—unless it is com-

*See John Burnet, *Greek philosophy. Thales to Plato* (London, 1932), pp. 320-322, 330; also R. E. Taylor, *Plato. The man and his work* (new ed., New York, 1936), pp. 505-506.

posed of points."* In this quotation it is evident that Aristotle had the arithmetical zero in mind, for it is regarded as bearing to number the same relationship as does a point to a line. Moreover, the impossibility of division by zero is here definitely stated almost fifteen hundred years before the time of Bhaskara.†

Unfortunately, the Greek interpretation of the word number was very limited. From the time of Thales "number" had been generally accepted as a collection or system of units—"a plurality of 'ones' and a certain quantity of them."‡ This definition included only the natural numbers. In fact, Aristotle went so far as to say that "the smallest number in the strict sense of the word 'number' is two."§ Although the Greeks constructed a sound and extensive theory of commensurable and incommensurable ratios, these were excluded from the realm of number. Needless to say, so also was zero. The *symbol* and the *concept* were familiar to them, but they never achieved the full status of *number*.

The Arabic notation is now recognized as misnamed. The Arabs were not its originators, but simply adopted the system and transmitted it to Europe during the Middle Ages. The source conventionally has been placed in India during the early centuries of our era, so that now the notation customarily is referred to as the Hindu-Arabic. However, the evidence in this connection is far from clear and the possibility of some other origin—perhaps in the Greek world—must be admitted. In any case, the so-called Hindu-Arabic system of numeration involves no principle not familiar to the world several thousand years ago. The ten-scale, local value, symbol for zero, and cipherization were widely used in antiquity. Notwithstanding ubiquitous categorical assertions to the contrary, the Hindus definitely were not the first inventors of any one of these fundamental aspects of numeration, although they may have been independent rediscoverers of one or more of them.

Unqualified claims* for the Indian origin of zero must necessarily be rejected. Nevertheless, it may well be that the Hindus first adapted

**Physica IV.* 8.215^b. The translation given is that of W. D. Ross and J. A. Smith (*The works of Aristotle*, 11 Vols., Oxford, 1908-1931, Vol. II) except that I have substituted the word zero for the symbol 0, inasmuch as there is no evidence that Aristotle here used a symbol.

†See my paper, "An early reference to division by zero," *American Mathematical Monthly*, L (1943), 487-491.

‡Aristotle, *Physica III.* 7.207^b.

§*Physica IV.* 12.220^a.

*Bibhutibhusan Datta has most erroneously said of zero that "the world is gradually adopting the view that the credit of the invention is entirely due to the Hindus." "Early literary evidence of the use of the zero in India," *American Mathematical Monthly*, XXXIII (1926), 449-454). In another place he has incorrectly added,

this to a *decimal* ciphered positional type of numeration. The Babylonians and Greeks had used zero only in the sexagesimal system, and the Mayas only in their quasi-vigesimal notation. Its adaptation to the ten-scale constituted a significant advance on the part of someone, probably during the early Christian centuries. The evidence in this respect still appears to point to the Hindus; but the situation is far from clear. The earliest undoubted occurrence of zero in India dates from 876, but its use may go back many centuries before this. The origin of this use is as uncertain as is the source of our numerals.* Considerable doubt remains also with respect to another point, the origin of our characteristic *form* of symbol for zero. It may have been a Hindu invention (as Hindu writers polemically maintain) or it may have been suggested by the Greek use of omicron (\omicron) for zero;† or it could have arisen in some other manner. The earliest form of the Hindu symbol for zero would appear to have been a dot, the present characteristic form being adopted later.‡ With respect to the source of our word "zero" the evidence is far more clear. The Hindus called it *sunya* (void), and this term passed into the Arabic as *sifr*. Fibonacci spoke of this as *zephiram*, which in the Italian of the following century took on various forms, including *cifra* and *zeron*, from which our words cipher and zero arose.§

At first the symbol for zero played the same role in Hindu numeration as it had earlier among the Babylonians, Mayas, and Greeks: it was a mark indicating an empty place in the positional notation. This is evidenced by the fact that, even in seventeenth-century Europe, serial representations of the ten Hindu-Arabic numerals generally placed the symbol for zero after the nine, in the *tenth* place, rather than before the one, as the *first* number. However, it was to be expected that, with the development of arithmetic, rules should be established for operating on the symbol zero as part of a number; and such rules ultimately led in turn to the recognition of zero as itself a number.

"The arithmetic of zero is entirely the Hindu contribution to the development of the mathematical science. With no other early nations do we find any treatment of zero."

"Early history of the arithmetic of zero and infinity in India," *Calcutta Mathematical Society Bulletin*, XVIII (1927), 165-176. As late as 1935 the claim of zero for the Hindus alone was reiterated. B. Datta and A. N. Singh, *History of Hindu Mathematics*. A source book. Part I. Numeral notation and arithmetic (Lahore, 1935). See p. 27.

*See Smith, *op. cit.*, II, 64-77; cf., however, Moritz Cantor, *Vorlesungen über Geschichte der Mathematik*, Vol. I (2nd ed., Leipzig, 1894), p. 563.

†Jules Sageret, "La genèse du zéro," *La Revue des Idées*, VII (1910), 320-340. See p. 339.

‡Florian Cajori, *A history of mathematics* (2nd ed., New York, 1931), pp. 88-89.

§Smith, *op. cit.*, II, 71-72. See also Gustav Oppert, "Ueber die Entstehung der Aera Dionysiana und den Ursprung der Null," *Berliner Gesellschaft für Anthropologie, Ethnologie und Urgeschichte*, Verhandlungen, 1900, pp. 102-136.

Aristotle had been familiar with the arithmetical aspect of zero, but the restrictive Greek definition of number excluded all but the positive integers. Precise definitions were not given by the Hindus, so that positive and negative, rational and irrational, quantities were included indiscriminately in the realm of number. Under such a situation there was nothing to stand in the way of accepting zero as a full-fledged number. The transition from the *symbol* zero to the *number* zero is not always clear, and the situation has been further obscured by inadvertencies on the part of some expositors. One of the foremost historians of Hindu mathematics has interpreted the reduction of

$$\frac{407150}{483920} \text{ to } \frac{40715}{48392}$$

as an operation on the *number* zero.* Although the *symbol* zero is indeed incidentally involved, the operation here concerns the integer ten, not the *number* zero. However, it appears that by 505 the Hindus may have looked upon zero as a number, for Varahamihira stated that the value of a quantity is unchanged if zero is added to or subtracted from it.† The evidence for the number zero becomes still more definite in 628 when Brahmagupta correctly stated that $0 \times (\neq a) = 0$, $0 \times 0 = 0$, and $\sqrt{0} = 0$. Brahmagupta expressed doubt as to $a \div 0$, the impossibility of which had been noted by Aristotle almost a thousand years before; and $0 \div 0$ he mistakenly thought was necessarily zero.‡ Bhaskara in 1114 characterized division by zero as infinity, although he gave no formal definition of this. The lack among the Hindus of that clear-cut reasoning which characterized Greek work is evident in Bhaskara's statements§ that $(a \times 0) \div 0 = a$ and

$$\frac{a}{0} + b = \frac{a}{0}.$$

Attempts have been made to justify such work in terms of zero as an infinitesimal and of infinity as a limiting value of an increasing sequence;¶ but they are unwarranted and misdirected. The fact that the Hindus calculated—whether correctly or incorrectly—with zero as

*Datta, "Early literary evidence of the use of the zero in India," *American Mathematical Monthly*, XXXVIII (1931), 566-572. See p. 567. Cf. pp. 568-569; also p. 570.

†Datta (1926), pp. 451-452.

‡Datta (1927), pp. 169-170. See also H. T. Colebrooke, *Algebra, with arithmetic and mensuration, from the Sanscrit of Brahmagupta and Bhaskara* (London, 1817), pp. 339-340; and S. R. Das, "The origin and development of numerals," *Indian Historical Quarterly*, III (1927), 97-120, 356-375, especially p. 119.

§Datta (1927), pp. 170-175. Cf. also Cantor, *op. cit.*, p. 576.

¶See Datta and Singh (1935), pp. 238-243.

with other numbers, indicates clearly however, that they looked upon zero as itself a number.

The Hindus did not define the domain of number, but it is fairly evident that it included what are now known as the real numbers, positive, negative, and zero. In this respect the Arabic and Latin medieval civilization did not follow them, although Al-Khowarizmi, Fibonacci (Leonardo of Pisa), Sacrobosco, Villedieu, Jordanus, and others popularized the Hindu numerals. Only positive roots of equations were recognized, and of these Al-Khowarizmi accepted only the rational. Roots of numbers which were not rational were referred to as surd, absurd, irregular, irrational, or inexplicable; negative numbers were sometimes called fictitious. Zero was not regarded as a root of an equation, and there is no evidence that it was thought of as a number by itself. In serial representation of the first ten digits it again habitually follows the 9 rather than precedes the 1. However, the early modern development of algebra seems to have led to the recognition once more of the *number* zero. This change is usually placed in the sixteenth century,* or even in the seventeenth.† However, one sees the rise of this attitude as early as 1484 in the *Triparty* of Chuquet. In mentioning zero he wrote that it "ne vault ou signifie rien . . . et pour ce est appellée chiffre ou nulle ou figure de nulle valeur."‡ This language does not necessarily imply that zero is to be regarded as a number, but this understanding is evident from Chuquet's use of it. He wrote $.0. m. 12$ for $0-12$ and $.0. p. 12.$ for $0+12$. Moreover, he appears to have been the first person to consider zero (as well as positive and negative integers) as an exponent. He wrote $.9.^{\circ}$ for $9x^{\circ}$, or 9; and in a table of powers (integral and zero) of two he has the number one corresponding to the exponent zero. Although Chuquet seems in general not to have admitted zero as a root of an equation, nevertheless in one such case he said the number sought was zero.§

Chuquet's work was not published until almost four centuries later, but it may have influenced his contemporaries through the circulation of his manuscripts. At any rate, the use of zero as a number was continued in the algebra of the sixteenth century. Stifel in 1544 wrote the polynomial x^3+1 in a form corresponding to x^3+0x^2+0x+1 ;

*See, for example, G. Eneström, "Über die Anfänge der Benutzung von Null als eine wirkliche Grösse," *Bibliotheca Mathematica* (3), VII (1906-1907), 309.

†J. Tropfke, *Geschichte der Elementar-Mathematik*, Vol. II, (2nd ed., Leipzig, 1921), pp. 7-9, 56.

‡Ch. Lambo, "Une algèbre française de 1484. Nicolas Chuquet," *Revue des Questions Scientifiques* (3), II (1902), 442-472. See p. 446. Or see Aristide Marre, "Notice sur Nicolas Chuquet et son Triparty en la science des nombres," *Bullettino di Bibliografia e di Storia delle Scienze Matematiche e Fisiche*, XIII (1880), 555-659, 693-814.

§Lambo, *op. cit.*, pp. 366-467.

Tartaglia in 1556 wrote numeral forms such as $\sqrt{45}+0$ and $\sqrt{45}-0$; Cardan in 1570 wrote the equivalents of $x^3=0+x$ and $x^3=216+0x$.^{*} These men appear not to have recognized zero as a root of an equation, but this step was taken in 1629 by Girard.[†] From this time on—two thousand years after the symbol and concept had found a place in numeration—there was general acceptance of zero as a full-fledged number. From the point of view of elementary mathematics the development may therefore be regarded as completed; but from a more advanced standpoint the history of zero might better be carried somewhat further. The critical reexamination of mathematical concepts carried out in the nineteenth century led Frege in 1884 to what may be regarded as the first satisfactory definition of cardinal number—the class of all equivalent classes. One is easily tempted to regard the class of all null classes as the final step in the rise of zero; but this brief survey will be concluded with a reminder—strengthened by the paradoxes of Russell and others—that the story of number in general, and of zero in particular, probably never will end. The endlessness of mathematical research is one of the brightest facets of civilization.

^{*}See Eneström, *loc. cit.*

[†]See *Encyclopédie des sciences mathématiques*, I, 1¹ (1904), p. 33, note 147.

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SOLUTIONS

No. 527. Proposed by *D. L. MacKay*, Evander Childs High School, New York.

Construct a triangle ABC , given a , $b+c$, and $(p-q)/h_a$, where p and q are the projections of the sides b and c upon the side a .

Solution by *Earl V. Greer*, Bethany-Peniel College, Bethany, Oklahoma.

Let $AD = h_a$, $AA' = m_a$, and $p-q : h_a = m : n$. Since $DA' = (p-q)/2$ and $DA' : AD = m : 2n$, the triangle ADA' is known in specie, and the angle $AA'D$ may be constructed.

Lay off $BC = a$, and at the mid-point A' construct an angle $XA'B$ equal to the angle $AA'D$. We now have to find a point A on $A'X$ such that $BA + AC$ shall be equal to the given segment $b+c$. This problem is known and can be solved with ruler and compass (see Altshiller-Court, *College Geometry*, Art. 356).

Also solved by *Frank Hawthorne* and the *Proposer*.

No. 534. Proposed by *N. A. Court*, University of Oklahoma.

The line joining the center of the twelve-point sphere of a tetrahedron (T) to the midpoint of the segment determined by the Monge point and a vertex of (T) passes through the corresponding vertex of the twin tetrahedron of (T).

State the corresponding proposition for the orthocentric tetrahedron.

Solution by *Earl V. Greer*, Bethany-Peniel College, Bethany, Oklahoma.

Let A and A' be corresponding vertices of (T) and its twin tetrahedron (T') , and let M and G respectively be the Monge point and centroid of (T) . Let P be the midpoint of the line AM . The lines $A'P$ and MG are medians of the triangle $AA'M$ and meet in a point L which divides the line MG in the ratio 2 : 1. Therefore L is the center of the twelve-point sphere of (T) , and the points P , L , and A' are collinear.

The corresponding proposition for the orthocentric tetrahedron is as follows: The line joining the center of the second twelve-point sphere of an orthocentric tetrahedron (T) to the midpoint of the segment determined by the orthocenter and a vertex of (T) passes through the corresponding vertex of the twin tetrahedron of (T) .

(References: See Articles 200, 231 and 764 of N. A. Court, *Modern Pure Solid Geometry*, Macmillan, 1935.)

Also solved by *L. M. Kelly* and the *Proposer*.

No. 536. Proposed by *E. Hoff*.

Within a time T , a train must pass a certain signal tower at a distance D . Its initial velocity is zero. Suppose the upper bound of acceleration is M'' and is fixed. Let M' be the upper bound of velocity. There is a lower bound to the values of M' which is greater than the average velocity D/T . Compute this lower bound for M' .

EDITORIAL NOTE. For a given distance to be traversed in a given time, the least top speed M' requires the greatest possible time spent at the speed M' . Let the train accelerate as quickly as possible to the speed M' and continue at that speed for the rest of the distance. Then we have $M' = M''t_1$, where t_1 is the length of the acceleration period. Hence $D_1 = \frac{1}{2}M''t_1^2$ is the distance traversed during the acceleration period. Since the rest of the journey is taken at the uniform speed M' , we have

$$\begin{aligned} T &= t_1 + (D - D_1)/M' = t_1 + (D - \frac{1}{2}M''t_1^2)/M' \\ &= \frac{M'}{M''} + \frac{2DM'' - M'^2}{2M'M''}, \end{aligned}$$

$$\text{or} \quad M'^2 - 2M''TM' + 2DM'' = 0,$$

$$\text{whence} \quad M' = M''T - (M''^2T^2 - 2DM'')^{\frac{1}{2}},$$

and this is the least possible value of M' .

It has been assumed that D does not exceed $\frac{1}{2}M''T^2$, otherwise it is impossible for the train to reach the tower in time regardless of speed limits. It is easy to verify that M' , as given above, is greater than D/T .—E. P. S.

No. 538. Proposed by *H. T. R. Aude*, Colgate University.

While working on a triangle problem a student noticed that the three given sides were represented by three relatively prime integers in arithmetic progression. He solved the problem and found that one of the angles was twice as large as another. Find the triangle.

Solution by *J. S. Guérin*, student, Catholic University of America, Washington, D. C.

The solution may be made to depend on the lemma: In a triangle ABC , if $\angle A = 2\angle B$, then $b(b+c) = a^2$.

Indeed, AD being the internal bisector of angle A , one has

$$(1) \quad AB \cdot AC = BD \cdot DC + AD^2 = BD(BD + DC) = BD \cdot BC$$

since $AD = BD$. But $AB/AC = BD/DC$ or $(AB+AC)/AB = BC/BD$. Hence $BD = AB \cdot BC / (AB+AC)$. Upon substituting this value in (1), one has the proof of the lemma.

The solution of the problem follows easily. Let $a, b = a-d, c = a+d$ be the three sides of the given triangle. Hence $\angle B < \angle A < \angle C$. There are three cases to be considered.

I. If $\angle C = 2\angle A$, then $a(a+b) = c^2$ or $a(2a-d) = (a+d)^2$ or

$$a = \frac{1}{2}d(3 + \sqrt{13}).$$

Thus a, b, c cannot all be integers.

II. If $\angle C = 2\angle B$, then $b(b+a) = c^2$ or $a^2 - 5ad = 0$. Thus one obtains $a = 5d, b = 4d, c = 6d$.

III. If $\angle A = 2\angle B$, then $b(b+c) = a^2$ or $a^2 - 2ad = 0$. Thus one has $a = 2d, b = d, c = 3d$, but this is not a triangle.

Finally it is clear that $a = 5, b = 4, c = 6$, obtained from II, provide the unique solution of the problem.

Also solved by *W. V. Parker* (using the relation $(s-a)/s = \tan \frac{1}{2}B \cdot \tan \frac{1}{2}C$), *A. E. Gault*, *Earl V. Greer*, *L. M. Kelly*, *Jerzy Szmojsz*, and the *Proposer*.

No. 542. Proposed by *E. Hoff*.

$$\text{Prove: } 4 \cos \frac{B-C}{3} \cos \frac{2C+A}{3} \cos \frac{A+2B}{3} = \cos(B-C),$$

if $A+B+C=2\pi$.

Solution by *Jessie E. Swanson*, Valparaiso University.

Applying a familiar trigonometric identity to the last two factors of the left member, we have

$$2 \cos \frac{B-C}{3} \left[\cos \frac{2A+2B+2C}{3} + \cos \frac{2B-2C}{3} \right].$$

Since $\cos[(2A+2B+2C)/3] = \cos(4\pi/3) = -\frac{1}{2}$, this result becomes

$$- \cos \frac{B-C}{3} + 2 \cos \frac{2B-2C}{3} \cos \frac{B-C}{3}.$$

A second application of the identity gives

$$- \cos \frac{B-C}{3} + \left[\cos \frac{3B-3C}{3} + \cos \frac{B-C}{3} \right],$$

which is identically $\cos(B-C)$ as required.

Also solved by *E. F. Allen*, *D. H. Erkiletian, Jr.*, *Henry E. Fettis*, *A. E. Gault*, *J. S. Guérin*, *Pvt. N. Kushta*, *D. L. MacKay*, *T. Tanimoto*, *J. Ernest Wilkins, Jr.*, and the *Proposer*.

No. 543. Proposed by *Nev. R. Mind*.

Determine the values of x which satisfy the relation

$$\cos(A-x)\cos(B-x)\cos(C-x) = \pm \cos(A+x)\cos(B+x)\cos(C+x)$$

assuming that $A+B+C=180^\circ$.

Solution by *J. S. Guérin*, student, Catholic University, Washington, D. C.

Upon substituting the addition formula for cosines and dividing the result by $\cos A \cdot \cos B \cdot \cos C \cdot \cos^3 x$, one easily reduces the given equation to

$$\begin{aligned} (1+\tan A \tan x)(1+\tan B \tan x)(1+\tan C \tan x) \\ = (1-\tan A \tan x)(1-\tan B \tan x)(1-\tan C \tan x). \end{aligned}$$

Consider first the relation in which the positive sign is taken. By performing the multiplications and combining the similar terms, one obtains

$$2\tan^2x(\tan A \cdot \tan B \cdot \tan C) + 2 \tan x(\tan A + \tan B + \tan C) = 0.$$

The two quantities in parentheses being equal, this becomes

$$\tan x(1 + \tan^2x) = 0.$$

Hence the only solutions are given by $x = k\pi$, where k is an integer or zero.

If the negative sign holds, reductions similar to those just used give

$$(1) \quad \tan^2x(\tan A \tan B + \tan B \tan C + \tan A \tan C) = -1.$$

Let x_1 be the principal value of $\arctan(-1/Q)^{1/2}$, where Q is the value of the quantity in parentheses in the last equation. Then

$$(2) \quad x = k\pi \pm x_1, \quad x_1 = \text{Arctan}(-1/Q)^{1/2}.$$

Obvious modification of this argument shows that the result is still valid for A, B or $C = 90^\circ$: Then $x_1 = 0$.

One may obtain another form for Q as follows:

$$\begin{aligned} Q &= \tan B \left(\frac{\sin A}{\cos A} + \frac{\sin C}{\cos C} \right) + \frac{\sin A \sin C}{\cos A \cos C} \\ &= \frac{\sin B}{\cos B} \left(\frac{\sin B}{\cos A \cos C} \right) + \frac{\cos B + \cos A \cos C}{\cos A \cos C} \\ &= \frac{(\sin^2 B + \cos^2 B) + \cos A \cos B \cos C}{\cos A \cos B \cos C} \\ &= \sec A \sec B \sec C + 1. \end{aligned}$$

Evidently x is not real if A, B and C are all acute angles.

EDITORIAL NOTE. The *Proposer* states the above results, using x_1 in (2) in the form $\text{Arcsin}(-\cos A \cos B \cos C)^{1/2}$. He also calls attention to the obvious root $x = (k + \frac{1}{2})\pi$. (This root was lost in the above solution because of dividing through by \cos^2x at the beginning.) Also solved by *D. L. MacKay* who obtains x_1 in the form

$$\text{Arccos}(\frac{1}{2} \sin^2 A + \frac{1}{2} \sin^2 B + \frac{1}{2} \sin^2 C)^{1/2}.$$

No. 545. Proposed by *E. P. Starke*, Rutgers University.

Let S be a positive number ($\neq 1$) such that S, S^2, S^4 is an arithmetic progression. Show that S is the length of a side of a regular decagon inscribed in the unit circle.

I. Solution by *E. V. Greer*, Bethany-Peniel College, Bethany, Oklahoma.

Let d represent the length of the side of a regular decagon inscribed in the unit circle. Draw radii to its extremities. Applying the law of sines and the law of cosines to the triangle thus formed, one obtains the relations

$$\begin{aligned} 1/d &= \sin 72^\circ / \sin 36^\circ = 2 \cos 36^\circ, \\ d^2 &= 2 - 2 \cos 36^\circ. \end{aligned}$$

Hence $d^2 - 2d + 1 = 0$. On the other hand, the hypothesis about S provides the relation $S^4 - S^2 = S^2 - S$ or, since $S \neq 0$, $S^3 - 2S + 1 = 0$.

In conclusion, both d and S are different from unity and are positive roots of the equation $x^3 - 2x + 1 = 0$ which has but one positive root other than $x = 1$. Therefore $d = S$.

II. Solution by *A. E. Meder, Jr.*, New Jersey College for Women.

The equation determining S is $S^4 - 2S^2 + S = 0$ or

$$(1) \quad S(S-1)(S^2+S-1) = 0.$$

The side of the decagon is $2 \sin 18^\circ$. Let $z = \cos 18^\circ + i \sin 18^\circ$, so that $z^{20} = 1$. Then

$$2 \sin 18^\circ = (z^{19} - z)i = (z^{19} - z)z^5 = z^4 - z^6.$$

$$\begin{aligned} \text{Therefore } 4 \sin^2 18^\circ + 2 \sin 18^\circ - 1 &= z^8 - 2z^{10} + z^{12} + z^4 - z^6 - 1 \\ &= 1 + z^4 + z^8 + z^{12} + z^{16} \\ &= (1 - z^{20}) / (1 - z^4) = 0, \end{aligned}$$

where it is to be noted that $z^{10} = -1$, $z^{16} = -z^6$, $z^4 \neq 1$.

Thus $2 \sin 18^\circ$ is a positive root ($\neq 1$) of (1), which proves the proposition.

III. Remarks by *Nev. R. Mind*.

The number $S = \frac{1}{2}(\sqrt{5} - 1)$ has many interesting properties, for example,

$$1/S = \frac{1}{2}(\sqrt{5} + 1) = \sqrt{1 + \sqrt{1 + \sqrt{1 + \dots}}} \text{ ad infinitum.}$$

(See *American Mathematical Monthly*, 1917, p. 32).

The number S satisfies the proportion

$$1 : S = S : (1 - S),$$

and is therefore the longer segment obtained by dividing the radius in extreme and mean ratio. This proportion has been called the "golden section" and the "divine proportion". It seems to have been used extensively by artists and architects both in antiquity and during the Renaissance, (Dürer, Leonardo da Vinci, etc.)

Others believe that this proportion is of frequent occurrence in nature (Matila C. Ghyka, *Esthétique des proportions*, Paris, 1927.)

Also solved by *E. F. Allen*, *F. M. Carpenter*, and *D. H. Erkiletian, Jr.*, *H. E. Fettis*, *A. E. Gault*, *J. S. Guérin*, *Pvt. William Leong*, *D. L. MacKay*, and *J. Ernest Wilkins, Jr.*

PROPOSALS

No. 567. Proposed by *Frank Hawthorne*, Allegheny College, Meadville, Pa.

An alley has two poles placed against bottoms of buildings on each side and resting against the opposite building, the plane of the figure being perpendicular to the road and to both walls. The length of the poles and the height of the point of intersection being given, obtain the equation which is satisfied by the width of the alley.

No. 568. Proposed by *Frank C. Gentry*, University of New Mexico.

The base of a variable triangle is fixed and the opposite vertex describes a line perpendicular to the line of the base. Show that the Euler line of the triangle is tangent to a fixed conic. Discuss the nature of the conic.

No. 569. Proposed by *Paul D. Thomas*, U. S. Navy.

Construct a triangle ABC , having given the inradius r , the median m_a , and the exradius r' relative to the side a .

No. 570. Proposed by *E. P. Starke*, Rutgers University.

A chain (perfectly flexible and inextensible) hangs with its ends fastened at two points, A , B , in a horizontal line. The lowest point of the chain is p units below the line AB , and its length is s units. Determine the distance AB in terms of p and s .

No. 571. Proposed by *Pvt. Nicholas Kushla*, University of Illinois.

Determine the limits of the following functions as $x \rightarrow \infty$

$$(a) \ e^x - x, \quad (b) \ x^2 - \csc^2(1/x).$$

No. 572. Proposed by *Julius S. Miller*, Dillard University, New Orleans.

A uniform chain of length L and total mass M is held vertically with its lower end just touching a platform balance. The fixed upper end is released and the chain "accumulates" on the scale pan. What is the maximum reading of the balance?

No. 573. Proposed by *Fred G. Fender*, Bloomfield, N. J.

A summer camper rowed upstream one mile when his hat blew off into the water beside him. As it was an old hat he decided to let it go. Ten minutes later he remembered that he had put his return ticket under the hatband. Rowing at the same rate as before, he just reached the hat (and the ticket) at the starting point. How fast was the stream flowing?

No. 574. Proposed by *N. A. Court*, University of Oklahoma.

A sphere having its center on the line of shortest distance of the two skew lines AB , CD meets these lines in the points A , B and C , D , respectively. Show that $AC = BD$ and $AD = BC$.

Bibliography and Reviews

Edited by

H. A. SIMMONS and P. K. SMITH

Geometry with Military and Naval Applications. By Willis F. Kern and James R. Bland. John Wiley and Sons, New York, 1943. vii + 152 pages.

This book presents the essentials of solid geometry in concise form. The introductory chapter (computation suggestions and plane mensuration), appendix A (systems of angle measurement, trigonometry of the right triangle), appendix B (Cavalieri's theorem, summary of theorems of plane geometry) and four place logarithmic tables form the point of departure for theory and calculation. Simpson's rule is also introduced.

The portion of the text devoted to solid geometry is divided into seven chapters as follows: 2, lines, planes, angles; 3, solids for which $V = Bh$; 4, solids for which $V = \frac{1}{3}Bh$; 5, solids for which $V = (\text{mean } B)h$; 6, the sphere; 7, volumes and surfaces of revolution, polyhedrons; 8, summary and review.

Simplicity and uniformity of attack, stress on applications (especially military and naval), plus numerous clear diagrams should make this presentation appeal to high school students and instructors alike.

Virginia Military Institute.

W. E. BYRNE.

Celestial Navigation, A Problem Manual. By Walter Hadel, McGraw-Hill, New York, 1944. 261 pages + xiii.

With its limited purpose, this book is an excellent one. Throughout, from page to page, it hews to the simple, direct line of giving precepts and examples for the practical navigator of an airplane. Separate sheets of answers accompany the book. It assumes a minimum of knowledge of astronomy and mathematics. Each of his chapters (which he calls Quiz No. 1, No. 2, etc.) begins with explanatory notes. These and the follow-up through each type of problem are very clear and make the book a good teaching manual.

Full excerpts from Commander Ageton's very convenient tables are given. Shorter samples of many other are included such as those from American Air Almanac and the justly famous Hydrographic Office No. 214.

Mr. Hadel has done a service to all who teach, learn or practice navigation at a time when briefness, speed and certainty of procedure are at a high premium.

Dearborn Observatory.

OLIVER J. LEE.

Navigation. By Lyman M. Kells, Willis F. Kern, and James R. Bland. McGraw-Hill Book Company, New York, 1943. xx + 479 pages.

Within the last fifteen months several new texts on air and surface navigation have come from the press. The text here reviewed is largely on surface navigation. In its pages the treatise most certainly shows the authors' thorough acquaintance with

those two long-time standards of navigation texts by Bowditch and Dutton. The text is not so extensive as Dutton; it possesses a freshness of style and clarity not attained in Dutton nor Bowditch. However, it does not replace either of these two classics in navigation.

Chapter I gives the basic trigonometry of navigation. Throughout the text the trigonometric numerical work is arranged in the boxed, compact form. This is so helpful in long computations.

Chapter II is devoted to instruments. The compass, pelorus, and sextant are treated too briefly. The optical principle of the sextant is not given, and the magnetic principles of deviation are covered only briefly.

"Charts" is the subject of Chapter IV. On this essential topic the text is outstanding. The chapter is supplemented by a critical, mathematical treatment of the Mercator's chart in Appendix D. This appendix is a pleasure to read.

"Piloting", "Navigation Aids", and "Ship and Plane Maneuvers" are treated in Chapters IV, V, and VI, respectively. The chapter on piloting is quite clear; however, the practice suggested of writing on the diagrams given in the text for solving certain piloting exercises might not be so good (especially for a V-12 student who must use the text following another student). The practice of pasting tracing paper on the figure and writing on the tracing paper has been found satisfactory. In the theory of lines of position by radio bearings it would have been quite helpful had an explanation been given for the choice of the E. P. in the case of three lines of position. "Ship and Plane Maneuvers" is very brief in comparison to the treatment by Dutton.

A "Review of Spherical Trigonometry" is the content of Chapter VII. In Chapter VIII the "Sailings and Nautical Astronomy" is the subject matter. This chapter is condensed but clearly and effectively written. The first eight pages briefly treat the sailings, and the remainder of the chapter lays down the principle of lines of position by Marcq St. Hilaire. In this chapter two pages from H. O. 214 are given and these tables explained. Ageton's Formulas are derived and a reference is made to "A Manual of Celestial Navigation" by Ageton in which Ageton's method (H. O. 211) is explained fully. In Rules A, B, and C, on page 253, the values of R , K , and h are not just "taken" as indicated. These signs and values on R , K , and h follow from formulas (4), (5), and (6), on page 253. The student might be led to think "take" here has the sense of assumption. At least a comment should be made on this situation. In consideration of Rule D, on page 253, the spherical triangle formed by N , P and the due west point of the horizon, in Figure 16, constitutes an area in which M may fall and $K-L$ be positive and still t and Z be in opposite quadrants. This case is not mentioned.

Chapters IX and X cover the troublesome subject of "Time" and explain the uses of the Nautical and Air Almanacs. If the student masters the contents of these chapters he has two basic procedures for finding the L. H. A. of a body. Most of the exercises requiring the Nautical Almanac are based on the 1942 issue. Excerpts are given in Chapter X from the pages of the Air Almanac.

The "Meridian Altitude" and the various corrections on the altitude of a body due to the ship's motion are treated in Chapter XI. The "Fix" and days work are the topics of Chapter XII. In this chapter numerous solutions of exercises are given at great length showing the practical procedure of the navigator's work at sea. Nautical "Astronomy and Star Identification" is taken up in Chapter XII. Appendices A, B, C, D, and E are devoted to "Logarithms", the "Range Finder", the "Hagner Planetarium", "Map Supplement", and forms for solving star sights, respectively.

This text is a genuine contribution to the problem of teaching navigation to the many prospective officers at this time. The format of the text is especially pleasing.

The numerous incorrect exercises make for some difficulty in the use of the text. Exercise 5(b), page 135, is not possible. The equation on page 150 should read

$C_m = C_c + D$. The statement, on page 152, concerning the use of Napier's diagram should be corrected as follows: interchange C_c and C_m in the first line, change C_m to C_c in the second line, and interchange C_m and C_c in the fifth line. In Problem 11, page 160, the positions of Pine Knoll and Atlas Island are given off of the small area plotting sheet which is to be made by directions in the text. The G. H. A. and L. H. A. in Fig. 13 and in Fig. 14 are not properly indicated. At the bottom of page 279 in the place of $23^{\circ} 56' 3.4''$ should appear $23^{\circ} 56' 4.1''$.

A corrected issue of this text has been announced. The text is recommended for the V-12 Program.

Louisiana Polytechnic Institute.

P. K. SMITH.

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